ANALYSIS EXAM BREAKDOWN

August 27, 2019

Contents

1. Overview

This is a collection of past comprehensive exams in analysis offered at Western.

1.1. Caveat emptor.

- The list of exams is *not* comprehensive:
  There are several gaps among old exams, and we do not intend to fill them.
- This document is likely to have mistakes:
  A team of us[1] typeset scans of the original exams, and we may have introduced typos.
- Not every problem was reproduced exactly from the original:
  We made occasional minor editorial changes, some of which are highlighted in blue.
- There are *no* solutions:
  The best way to study is to write your own solutions.
- The exams are in the order in which the reader is intended to work through them:
  Upcoming exams are more likely to resemble recent exams than old exams.

1.2. Features.

- In most problem statements, many key terms are highlighted in magenta.
- An index of common concepts appears at the end.

If you have comments or would like to contribute to the document, please contact Chris Hall (`chall69@uwo.ca`).

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2. Spring 2019 (May)

1. Let z be a complex variable and set $f(z) = \sum_{n\geq 0} c_n z^n$ where the coefficients are the Fibonacci numbers defined by $c_0 = c_1 = 1$ and $c_{n+2} = c_{n+1} + c_n$.

   (a) Show that $f(z) = \frac{1}{1-z-z^2}$ on any disc $D(0,R)$ on which the series converges.

   (b) Find the radius of convergence of the series.

2. Suppose $f\colon [0,1] \to \mathbb{R}$ is continuous, and $a, b \in \mathbb{R}$. Prove that the boundary value equation
$$u^g = f, \quad u(0) = a, \quad u(1) = b$$
has a unique solution $u \in C^2([0,1])$ given by
$$u(x) = (1-x)\left(a - \int_0^x tf(t)\,dt\right) + x\left(b - \int_x^1 (1-t)f(t)\,dt\right).$$

3. Evaluate $\int_0^{2\pi} \frac{\sin^2\theta}{5+4\cos\theta} d\theta$.

4. Let d_1 and d_2 be metrics on a set X. Consider the following statements:

   P: For any metric space (Y,ρ) and any continuous $f\colon (X,d_1) \to (Y,\rho)$, the function $f\colon (X,d_2) \to (Y,\rho)$ is also continuous.

   Q: For any metric space (Y,ρ) and any continuous $f\colon (Y,\rho) \to (X,d_2)$, the function $f\colon (Y,\rho) \to (X,d_1)$ is also continuous.

   Prove that P is true if and only if Q is true.

5. Suppose f is holomorphic in a disc centered at the origin, that $f(0) = 0$, and that $f'(0) \neq 0$. From the inverse function theorem, we know f has an inverse g defined in some neighbourhood of 0. Show that on some open disc $D(0,\varepsilon)$, g is given by
$$g(z) = \frac{1}{2\pi i}\int_{|w|=\varepsilon} \frac{w\,f'(w)}{f(w)-z} dw.$$

6. Let $\varphi\colon \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function such that $\varphi(x) = 0$ when $|x| \geq 1$. Let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous. For each $x \in \mathbb{R}$, define $g(x) = \int_{-\infty}^{\infty} f(x-t)\varphi(t)dt$. Prove that g is differentiable on $\mathbb{R}$.

7. Let f be a holomorphic differentiable function defined on the open unit disc. Suppose there exists an open arc R on the unit circle having the property that $\lim f(z) = 1$, as z approaches R (z is in the open unit disc.) Prove that f is identically 1.

8. Let d be a positive integer and consider the sequence $(f_n)_{n=1}^{\infty}$, where $f_n\colon \mathbb{R}^d \to [0,1]$ for $n = 1, 2, \ldots$. Prove that there is a subsequence $(f_{n_k})_{k=1}^{\infty}$ such that for each $q \in \mathbb{Q}^d$, $\lim_{k\to\infty} f_{n_k}(q)$ converges.

3. Fall 2018 (October)

1. Suppose $f(x)$ is a function continuous on $[0,1]$ and differentiable on $(0,1)$. Suppose that $f(0)=0$ and $\int_0^1 f(x)dx = 1$. Prove that there exists a point $x_0 \in (0,1)$ such that $f'(x_0) > 1$.

2. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a non-constant function. A number $c \in \mathbb{R}$ is called a *period* of f, if $f(x+c) = f(x)$ for all $x \in \mathbb{R}$. The function f is then called *periodic* if there exists $p > 0$ such that $f(x+p) = f(x)$ for all $x \in \mathbb{R}$.

   (a) Show that the set of all periods of a given function f forms a subgroup of $(\mathbb{R}, +)$.

   (b) Give an example of a non-constant function f for which the group of periods is not discrete.

   (c) Prove that if f is non-constant and continuous, then the group of periods is discrete.

3. Suppose that f is a real-valued function defined on an open subset $\Omega \subset \mathbb{R}^n$ and that the partial derivatives $\partial f / \partial x_j$ exist and are bounded for $j = 1, \dots, n$. Prove that f is continuous on Ω.

4. Let X denote the space of all sequences of real numbers. Let $x = (x_k)_{k=1}^{\infty}$ and $y = (y_k)_{k=1}^{\infty}$ be arbitrary elements of X. Define
$$d(x,y) = \sum_{k=1}^{\infty} \frac{1}{k^2} \min\{|x_k - y_k|, 1\}.$$
Prove that (X, d) is a metric space.

5. Prove or give a counterexample to the following statement:
There is no non-zero polynomial $P(z)$ such that $P(z) \cdot e^{1/z}$ is an entire function.

6. Let γ be the curve $r = \frac{3}{2} + 3\cos\theta$, $\theta \in \mathbb{R}$, traversed one time counterclockwise. Evaluate the integral
$$\int_\gamma \frac{\sin(\pi z)}{z^2 - 3z + 2} dz.$$

7. Let $f(z) = \frac{1}{z} - \frac{1}{z^2+1}$. Find all possible Laurent expansions of f about $z_0 = i$ and determine where each is valid.

8. Let $(z_n)_{n=1}^{\infty}$ be a sequence of distinct complex numbers such that the series $\sum_{n=1}^{\infty} \frac{1}{|z_n|^3}$ converges, and let
$$f(z) = \sum_{n=1}^{\infty} \left(\frac{1}{(z - z_n)^2} - \frac{1}{z_n^2} \right).$$
Prove that f is meromorphic on $\mathbb{C}$ and find all its poles.

4. Spring 2018 (May)

1. Suppose f is a holomorphic function on a neighborhood of the closed unit disk, such that $|f(z)| \geq 2$ on the unit circle and $f(0) = 1$. Show that f has a zero in the unit disk.

2. Let C be the circle in the complex plane that has radius 3 and centre 0, traced once in the counter-clockwise direction. Calculate
$$\int_C \frac{e^z}{z^4 + z^2}\, dz.$$

3. Let f be a continuous function on $\mathbb{C}$ and holomorphic on $\mathbb{C} - \{z \in \mathbb{C} \mid \operatorname{Re} z = 0\}$. Prove that f is entire.

4. How many zeroes does the function $f(z) = \frac{1}{10}e^z - z$ have in the annulus $1 < |z| < 2$?

5. Show that the following limit exists:
$$\lim_{x \to \infty} \int_0^x \cos(t^3 + t)\, dt.$$

6. How many terms of the series
$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$
do you have to add in order to approximates its value to within $\frac{1}{10}$? (No need to give the best possible answer.)

7. Let $0 \leq \alpha < 1$ be a constant. Consider the function $f \colon \mathbb{R} \to \mathbb{R}$ defined by
$$f(x) = x - \alpha \sin x, \qquad x \in \mathbb{R}.$$
Show that f is one to one and onto and its inverse function is smooth.

8. Let $\Delta = \{(p_1, \ldots, p_n) \in \mathbb{R}^n : p_i \geq 0, \sum p_i = 1\}$. Consider the function $S \colon \Delta \to \mathbb{R}$ defined by
$$S(p_1, \ldots, p_n) = -\sum_{i=1}^{n} p_i \ln p_i.$$
(We define $0 \ln 0 = 0$.) What is the maximum value of S and where is it obtained? Prove your claim.

5. Fall 2017 (October)

1. Suppose $u, v \in (0,1)$.
   (a) Prove that x and y are well defined as functions of (u,v) by
   $$\sin(ux) = v, \quad 0 < ux < \pi/2, \qquad \text{and} \qquad \sin(uy) = v, \quad \pi/2 < uy < \pi.$$
   (b) Sketch the following three subsets of $\mathbb{R}^2$ on the same (large, clearly labeled) xy-axes:
   $$\begin{aligned} A &= \{(x,y) : u = 1/2, v \in (0,1)\}, \\ B &= \{(x,y) : u \in (0,1), v = 1/2\}, \text{ and} \\ C &= \{(x,y) : u, v \in (0,1)\}. \end{aligned}$$

2. Let $p(z)$ be the polynomial $az^n + z + 1$ with $n \geq 2$ and $a \in \mathbb{C}$.
   (a) Suppose $|a| < 1/2^n$. Prove that p has exactly one root in the disc $|z| < 2$.
   (b) Show that for any $a \in \mathbb{C}$, p has at least one root in the disc $|z| \leq 2$.

3. Let $x_1, x_2, \ldots$ be a sequence of real numbers and define $f\colon \mathbb{R} \to \mathbb{R}$ by $f(x) = \inf_{k=1,2,\ldots} k|x - x_k|$.
   (a) Show that if the sequence $\{x_n\}_{n=1}^{\infty}$ has no convergent subsequence then f is continuous.
   (b) Find a sequence $x_1, x_2, \ldots$ such that f is not continuous.

4. Let k, m, M be positive constants.
   (a) Suppose F is a continuous function satisfying $|F(Re^{it})| \leq \frac{M}{R^k}$ when $R > 0$ and $0 \leq t \leq \pi$. Prove:
   $$\lim_{R\to\infty} \int_{\Gamma} e^{imz} F(z)\, dz = 0$$
   where Γ is the semicircular arc $\{Re^{it} : 0 \leq t \leq \pi\}$.
   (b) Show that $\int_0^{\infty} \frac{\cos(mx)}{x^2+1} dx = \frac{\pi}{2} e^{-m}$.

5. Let (M,d) be a metric space and let X be the collection of all Cauchy sequences in M. For $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$ in X let $A(x,y) = (x_!, y_1, x_2, y_2, x_3, y_3, \ldots, \ldots)$. We say $x \sim y$ provided $A(x,y) \in X$.
   (a) Prove that $\sim$ is an equivalence relation on X.
   (b) Fix $x \in X$ and $m \in M$ and let y be the constant sequence $(m, m, m, \ldots)$. Show that x converges to m if and only if $x \sim y$.

6. Let Ω be a connected open subset of $\mathbb{C}$ and $f\colon \Omega \to \Omega$ be a holomorphic map such that $f \circ f = f$. Show that either f is the identity map on Ω or f is constant.

7. For each positive integer n, define $f_n\colon (0,\infty) \to \mathbb{R}$ by $f_n(x) = \int_0^1 t^{x-1}(1-t)^{n-1} dt$.
   (a) Prove that for each $x > 0$, $\lim_{n\to\infty} f_n(x) = 0$.
   (b) Prove that $\{f_n\}_{n=1}^{\infty}$ does not converge uniformly to the zero function on $(0,\infty)$.

6. Spring 2017 (May)

6.1. Real Analysis.

1. Let U be an open neighbourhood of $0 \in \mathbb{R}^n$, and let $f\colon U \to \mathbb{R}^n$ be Lipschitz continuous, with Lipschitz constant $K > 0$. Let $0 < a < 1$ be such that the closed ball $\overline{B}_{2a}(0)$ is contained in U and the norm $\|f(x)\| \le L$ for some constant $L > 0$ and all $x \in \overline{B}_{2a}(0)$. Let $b > 0$ be such that $b < \min\left\{\frac{a}{L}, \frac{1}{K}\right\}$.
   (a) For a point $x \in \overline{B}_a(0)$, let M_x be the set of continuous maps $\alpha\colon [-b, b] \to \overline{B}_{2a}(0)$ satisfying $\alpha(0) = x$, and for each map $\alpha \in M_x$, define $S_x(\alpha)$ to be the map
   $$[-b, b] \ni t \mapsto x + \int_0^1 f(\alpha(u))\,\mathrm{d}u \in \mathbb{R}^n.$$
   Show that S_x maps M_x into M_x, and it is a contraction.
   (b) Show that, for every $x_0 \in \overline{B}_a(0)$, there exists a unique $\alpha_0 \in M_{x_0}$ satisfying
   $$\alpha_0(t) = x_0 + \int_0^t f(\alpha_0(u))\,\mathrm{d}u$$
   (i.e., a unique local solution to the initial value problem $x'(t) = f(x(t)), x(0) = x_0$).

2. Let $\Phi\colon (\mathbb{R}^n)^n \to \mathbb{R}$ be given as $\Phi(v_1, \dots, v_n) = \det[v_j^i]_{i,j=1,\dots,n}$, where $v_j = (v_j^1, \dots, v_j^n) \in \mathbb{R}^n$ for $j = 1, \dots, n$. Let $\{e_1, \dots, e_n\}$ be the standard orthonormal basis in $\mathbb{R}^n$, and let $h_j = (j, \dots, j) \in \mathbb{R}^n$ for $j = 1, \dots, n$. Evaluate $D\Phi(e_1, \dots, e_n).(h_1, \dots, h_n)$. Justify your answer.

3. Let $D = \{(x, y, z) \in \mathbb{R}^3 : 1 \le x^2 + y^2 + z^2 \le 4\}$. Evaluate the following, and justify your answer:
$$\int_D \sin(xy) - \sin(xz) + \sin(yz)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

4. (a) Show that the area of a planar region delimited by a closed simple smooth curve C is given by
   $$\frac{1}{2}\int_C x\,\mathrm{d}y - y\,\mathrm{d}x.$$
   (b) Compute $\int_C (xy - y^2)\mathrm{d}x + (x^2 + 3xy)\mathrm{d}y$, where C is the boundary of the bounded region delimited by the graphs of $y = x^3$ and $x = y^2$.

6.2. Complex Analysis.

5. Evaluate the following integral
$$\int_{\partial\Omega} \frac{e^{\pi z}}{2z^2 - i}\,dz,$$
where the domain Ω is given by
$$\Omega = \{z \in \mathbb{C} : |z| < 1,\ \operatorname{Im} z > 0,\ \operatorname{Re} z > 0\}\}.$$

6. Let $f(z)$ be an entire function that satisfies the inequality
$$|f(z)| \le A + B|z|^k, \qquad z \in \mathbb{C},$$
where $A, B > 0$ and k is a positive integer. Prove that $f(z)$ is a polynomial of degree at most k.

7. Suppose that $g(z)$ is a function that is holomorphic in the open unit disc $\mathbb{D} = \{z : |z| < 1\}$ and continuous on its closure $\bar{\mathbb{D}}$. Assume that $\operatorname{Im} g(z) \equiv 0$ on the unit circle $\partial\mathbb{D}$. Prove that $g(z)$ is a constant function.

8. Suppose that a sequence of holomorphic functions $f_n(z)$ on a domain $\Omega \subset \mathbb{C}$ converges to a function $f(z)$ uniformly on compacts in Ω. Prove that $f(z)$ is also a holomorphic function on Ω.

7. Fall 2016 (October)

1. Show that for any smooth function $f\colon \mathbb{R}^3 \to \mathbb{R}$ with *compact support* we have
$$\int_{\mathbb{R}^3} \frac{\Delta f(x)}{|x|}\, d^3x = 4\pi f(0),$$
where $\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$ is the Laplace operator.

   (Hint: Try to use *Green's second identity*: for any compact domain U with smooth boundary ∂U and smooth functions f, g on U, we have
$$\int_U (f\Delta g - g\Delta f)\, dv = \int_{\partial U} \left(f\frac{\partial g}{\partial \mathbf{n}} - g\frac{\partial f}{\partial \mathbf{n}} \right) ds,$$
where $\frac{\partial f}{\partial \mathbf{n}} = \nabla f \cdot \mathbf{n}$ is the directional derivative of f along the unit normal vector to the boundary of U.)

2. Let $u\colon [-1,1] \to \mathbb{R}$ be a smooth function such that $u(1) = u(-1) = 0$. Show that
$$\int_{-1}^{+1} u^2(s)\, ds \le 4 \int_{-1}^{+1} (u'(s))^2\, ds.$$
(Hint: Using the fundamental theorem of calculus, write $u(s) = \int_{-1}^{s} u'(t)\, dt$. Then try to estimate the latter integral using the Cauchy-Schwartz inequality $\int (fg) \le (\int f^2)^{1/2} (\int g^2)^{1/2}$.)

3. Let $\theta \in \mathbb{R}$ be an *irrational number*. Prove the following:
   (a) The set of numbers $\{n\theta \bmod 1 : n = 1, 2, 3, \ldots\}$ is dense in the interval $[0,1]$.
   (b) For any *finite* subset $K \subset \mathbb{Z}$ and any periodic function of the form $f(x) = \sum_{k\in K} a_k e^{2\pi i k x}$
$$\lim_{n\to\infty} \frac{1}{n+1} \left(\sum_{m=0}^{n} f(m\theta) \right) = \int_0^1 f(x)\, dx.$$

4. For $n = 1, 2, \ldots$, let
$$\gamma_n = \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) - \ln n.$$
Show that $\lim_{n\to\infty} \gamma_n$ exists. The limit is known as *Euler's constant* γ.

   (Hint: Show that the sequence is positive and decreasing.)

5. How many zeros does the polynomial $p(z) = z^7 + z^4 + 5z^3 + 1$ have in the annulus $1 < |z| < 2$?

6. Evaluate the integral:
$$\int_0^{2\pi} \frac{d\varphi}{1 - e^{\frac{\pi i}{2}} \cos\varphi + \frac{1}{4} e^{\frac{2\pi i}{7}}}.$$

7. Prove: if $f(z)$ is an entire function such that $|f(z)| \cdot |\mathrm{Im}(z)|^2 \le 1$, then $f(z) \equiv 0$.

8. Let $U \subset \mathbb{C}$ be an open subset, $z_0 \in U$, and $V = U - \{z_0\}$. Suppose $f\colon V \to \mathbb{C}$ is a continuous function such that $\int_\Gamma f(z)\, dz = 0$ whenever Γ is the boundary of a closed rectangle in V.

   Claim: *There is a holomorphic function $g\colon U \to \mathbb{C}$ such that $f(z) = g(z)$ for all $z \in V$.*

   Either prove the claim or state that it is not true and provide a counterexample.

8. Spring 2016 (May)

1. (a) Are the functions

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} \quad \text{and} \quad g(z) = \sum_{n=0}^{\infty} \frac{(z-i)^n}{(2-i)^{n+1}}$$

analytic continuations of each other? Justify your answer.

(b) The series

$$h(z) = \sum_{k=0}^{\infty} z^{2^k} = z + z^2 + z^4 + \cdots$$

converges for $|z| < 1$. (There is no need to prove this.) Show that it cannot be continued analytically beyond the unit disc.

2. Let (X, d) and (Y, ρ) be complete metric spaces. Prove that if A is a dense subset of X and $f\colon A \to Y$ is an isometry, then f extends to an isometry $F\colon X \to Y$. Give a clear definition of F, prove that it is well defined, and show it is an isometry.

3. Evaluate $\int_0^{2\pi} \frac{\sin^2\theta}{5+4\cos\theta}\, d\theta$.

4. Suppose $f\colon \mathbb{R}^2 \to \mathbb{R}$ is continuous. Prove that

$$g(t) = \int_0^t f(t,x)\, dx$$

defines g as a continuous function from $\mathbb{R}$ to $\mathbb{R}$.

5. Find all solutions of the equation $e^z = 1 + 2z$ satisfying $|z| < 1$.

6. Suppose $y = y(x)$ is the unique solution to the initial value problem

$$y'' = xy' + 3y, \quad y(0) = 0,\ y'(0) = 1.$$

(a) Prove that y and all of its derivatives are increasing functions on $[0,1]$.

(b) Express $y^{(5)}(1)$ in terms of $y'(1)$ and $y(1)$.

(c) Use Taylor's theorem to show that, for all $x \in [0,1]$,

$$\left| y(x) - x - \frac{2}{3}x^3 \right| \le \frac{1}{3}y'(1) + \frac{3}{10}y(1).$$

(In this question there is no need to give a closed form for the solution $y(x)$.)

7. Let $d_1, d_2, \ldots$ be metrics on X satisfying $d_k(x,y) \le 1$, for each k and all $x, y \in X$. Define d by

$$d(x,y) = \sum_{k=1}^{\infty} 2^{-k} d_k(x,y).$$

Then d is also a metric on X. (There is no need to prove this.) Show that if the metric space (X, d_k) is compact for each k, then the metric space (X, d) is also compact.

8. Let $h(t,z)$ be a continuous complex-valued function defined for $t \in [a,b]$ and $z \in \mathbb{C}$. Suppose, for each fixed t, that $h(t,z)$ is analytic. Show that

$$H(z) = \int_a^b h(t,z)\, dt$$

is an entire function.

9. Fall 2015 (October)

1. (a) Define Lipschitz continuous functions on the interval $I = [-1, 1]$.

   (b) Show that the uniform limit of Lipschitz continuous functions on the interval I may not be Lipschitz continuous.

   (c) Show that if a sequence $(f_m(x))_{m=1}^{\infty}$ converges uniformly on I, and all $f_m(x)$ are Lipschitz continuous with a uniform constant K, then the limit is also Lipschitz continuous with the constant K.

2. Let $f(x)$ be a function defined on $(0, 1]$ such that f is Riemann integrable on $[c, 1]$ for any $0 < c < 1$. Define
$$\int_0^1 f(x)\,dx = \lim_{c\to 0^+} \int_c^1 f(x)\,dx.$$

   (a) Show that if $f(x)$ is Riemann integrable on $[0, 1]$, then the standard definition of $\int_0^1 f(x)\,dx$ using Riemann sums and the definition given above agree.

   (b) Give an example of a function which is not Riemann integrable on $[0, 1]$ but for which the above limit exists.

3. Consider the function
$$F(x) = \begin{cases} \frac{xy^3}{x^2+y^4}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0). \end{cases}$$

   (a) Show that $F(x, y)$ is continuous at the origin.

   (b) Define what it means for a function $f(x, y)$ to be differentiable at $(0, 0)$.

   (c) Show that $F(x, y)$ above is not differentiable at the origin.

4. Let $I \subset \mathbb{R}$ be the open subinterval $(-1, 1)$ and $\mathcal{S}$ be the space of bounded continuous functions on I. Define
$$\rho(f, g) = \sup_{x\in I} |f(x) - g(x)|, \quad f, g \in \mathcal{S}.$$

   (a) Prove that ρ is a metric on $\mathcal{S}$ and that $(\mathcal{S}, \rho)$ is a complete metric space.

   (b) Prove that the map $H\colon (\mathcal{S}, \rho) \to \mathbb{R}$, given by $H(f) = f(0)$, is continuous.

5. Is there a polynomial $P(z)$ such that $P(z) \cdot e^{1/z}$ is an entire function? Justify your answer (i.e., give an example or prove it does not exist).

6. Evaluate
$$\int_{-\infty}^{\infty} \frac{x^2}{1 + x^4}\,dx.$$

7. Let f be a non-constant entire function. Show that the image of f is dense in $\mathbb{C}$.

8. Let $(z_n)_{n=1}^{\infty}$ be a sequence of distinct complex numbers such that the series $\sum_{n=1}^{\infty} \frac{1}{|z_n|^3}$ converges, and let
$$f(x) = \sum_{n=1}^{\infty} \left(\frac{1}{(z - z_n)^2} - \frac{1}{z_n^2} \right).$$
Prove that f is meromorphic on $\mathbb{C}$ and find all its poles.

10. Spring 2015 (May)

1. Find the number of roots of the polynomial $z^9 - 6z^4 + 3z - 1$ in $|z| < 1$.
2. Evaluate the integral:
$$\int_{|z|=4} \frac{e^{\frac{1}{z-1}}}{z-2}\, dz.$$
3. Does there exist an entire function $f(z)$ such that
$$f\left(\frac{1}{n}\right) = f\left(-\frac{1}{n}\right) = \frac{1}{n^{2015}}$$
for $n = 1, 2, 3, \ldots$?
4. For which $z \in \mathbb{C}$ does the following series converge?
$$\sum_{n=1}^{\infty} \left(\frac{z^n}{(n+1)!} + \frac{n}{z^n}\right)$$
5. Let A be the positive definite real n-by-n matrix. Show that
$$\int_{\mathbb{R}^n} e^{-X^t A X}\, dX = \frac{\pi^{n/2}}{\sqrt{\det A}}.$$
(You can use the fact that $\int_{\mathbb{R}} e^{-x^2}\, dx = \sqrt{\pi}$. Reduce the problem to 1-dimensional integrals. First work out the case where A is diagonal.)
6. Let $f \colon \mathbb{R} \to \mathbb{C}$ be a 2π-periodic *infinitely differentiable* function. For $n \in \mathbb{Z}$, let
$$\hat{f}(n) = \int_0^{2\pi} f(x) e^{-inx}\, dx$$
be the n-th *Fourier coefficient* of f. Show that $\hat{f}(n)$ is *rapid decay*, that is, for any positive integer k,
$$\left| n^k \hat{f}(n) \right| \to 0 \quad \text{as } n \to \infty.$$
(Hint: You can use the Riemann-Lebesgue lemma: for any L^1-function g, $|\hat{g}(n)| \to 0$ as $|n| \to \infty$. Use integration by parts.)
7. State the *inverse function theorem*. Give an example of a map $f \colon \mathbb{R}^2 \to \mathbb{R}^2$ such that for all (x, y), $Jf(x, y) \neq 0$, but f is not injective. (Jf denotes the Jacobian of f.)
8. (a) Use the *divergence theorem* to show that for any bounded domain $U \subset \mathbb{R}^2$ with smooth boundary ∂U and smooth functions φ, ψ on Ω, we have (Green's second identity)
$$\int_U (\psi \Delta\varphi - \varphi \Delta\psi)\, dx\, dy = \oint_{\partial U} \left(\psi \frac{\partial\phi}{\partial \mathbf{n}} - \varphi \frac{\partial\psi}{\partial \mathbf{n}}\right) dl,$$
where $\frac{\partial\phi}{\partial \mathbf{n}}$ is the directional derivative of φ in the direction of the outward pointing normal to the boundary, and $\Delta\varphi = \varphi_{xx} + \varphi_{yy}$ is the Laplacian of φ.

   (b) Use the above (Green's second) identity to show that for any smooth function with compact support $f \colon \mathbb{R}^2 \to \mathbb{R}$ we have
$$\int_{\mathbb{R}^2} (\ln r) \Delta f = -2\pi f(0)$$
where $r = \sqrt{x^2 + y^2}$.

11. Fall 2014 (October)

1. Show that
$$\int_{|z|=1} e^{\sin(1/z)} dz = 2\pi i.$$

2. Let (X, d) be a non-empty metric space such that, for each $x \in X$, the closed unit ball
$$B_x = \{y \in X : d(x, y) \leq 1\}$$
centred at x is compact. Prove that (X, d) is complete.

3. Evaluate
$$\int_{-\infty}^{\infty} \frac{dx}{1 + x^4}.$$

4. Show that every solution $y = y(x)$, to the differential equation
$$y''' + 2y'' - 2y = xe^x$$
is also a solution to the differential equation
$$y^{(5)} - 3y''' + 4y' - 2y = 0.$$

5. Let
$$f(z) = \frac{1}{z} - \frac{1}{(z+1)^2}.$$
Find all possible Laurent expansions of f about $z_0 = 1$ and determine where each is valid.

6. For each $(u, v) \in (0, \infty)^2$, define $F(u, v) = (v(1 + u)u, v(1 + u)/u)$. On large, clearly labeled axes, sketch the region $F^{-1}((0, 1)^2)$ and clearly identify the curve $F^{-1}(\{(x, x) : 0 < x < 1\})$. Make the change of variables $(x, y) = F(u, v)$ to evaluate
$$\int_0^1 \int_0^y \frac{dx\, dy}{y + \sqrt{xy}}.$$
[compare #6 of Fall 1997]

7. Show that there exists no function f analytic in the open unit disc $\{z : |z| < 1\}$ with the property that $|f(z)| \to \infty$ as $|z|$ increases to 1.

8. Suppose that $f_n \colon [0, \infty) \to \mathbb{R}$, for $n = 1, 2, \ldots$, are continuous functions such that
$$\lim_{n\to\infty} \int_0^n |f_n(t)| dt = 0.$$
For $x \geq 0$, let
$$F_n(x) = \int_0^x f_n(t) dt, \quad n = 1, 2, \ldots.$$
Prove that F_n converges uniformly to 0 on $[0, m]$ for every $m > 0$.

12. Spring 2014 (May)

1. (a) Show that for all $x \in \mathbb{R}$
$$f(x) := \left(\int_0^x e^{-t^2}\, dt\right)^2 + \int_0^1 \frac{e^{-x^2(t^2+1)}}{t^2+1}\, dt = \frac{\pi}{4}.$$
   (b) Deduce that
$$\int_0^\infty e^{-t^2}\, dt = \frac{\sqrt{\pi}}{2}.$$

2. Given $x \in \mathbb{R}$, $x > 0$, let $\langle\, x\,\rangle \in [0,1)$ be the fractional part of x. For $n \in \mathbb{N}$, define $f_n(x) = \langle\, nx\,\rangle$ and consider the series $f(x) = \sum_{n\geq 1} \frac{f_n(x)}{n^2}$.
   (a) Show that f converges uniformly on $\mathbb{R}$.
   (b) For a fixed n, find the discontinuities of the function $x \mapsto f_n(x)$ by computing the one-sided limits of the function.
   (c) Show that f is continuous at any irrational number.
   (d) Show that f is not continuous at any rational number.
   (e) Show that f is Riemann integrable on any bounded interval.

3. Let $X : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field
$$(2xz^3 + 6y)\mathbf{i} + (6x - 2yz)\mathbf{j} + (3x^2z^2 - y^2)\mathbf{k}.$$
Compute the line integral $\int_C X \cot r\, dr$ from $(1,-1,1)$ to $(2,1,-1)$ where C is the curve
$$C(t) = (2 - \cos(\pi t), 1 - 2\cos(\pi t), 1 - 8t^2).$$

4. Let C be the set of continuous real-valued functions on $[0,1]$. Given $f, g \in C$, define
$$d(f,g) = \sup_{x\in[0,1]} |f(x) - g(x)| \text{ and } \rho(f,g) = \int_0^1 |f(t) - g(t)|\, dt.$$
   (a) Show that d and ρ are metrics on C.
   (b) Prove that (C,d) is complete.
   (c) Prove the identity map $\mathrm{id}\colon (C,d) \to (C,\rho)$ is continuous.
   (d) Is the identity map $\mathrm{id}\colon (C,\rho) \to (C,d)$ is a homeomorphism? Explain.
   (e) Show that (C,ρ) is not complete.

5. Compute the integral
$$\int_{|z|=1/2} \frac{e^{1/z}}{1-z}\, dz.$$

6. Let f be a holomorphic function on a disc $U_R = \{z \in \mathbb{C}\colon |z| < R\}$ with $f(0) = 0$ and $|f(z)| < M$, for all $z \in U_R$.
   (a) Prove that $|f(z)| \leq \frac{M}{R}|z|$ in U_R, and $|f'(0)| \leq \frac{M}{R}$.
   (b) Show that the equality $|f'(0)| = \frac{M}{R}$ holds only if $f(z) = e^{i\alpha}\frac{M}{R}z$.

7. Prove that if two functions $f_1(z)$ and $f_2(z)$ are holomorphic in a domain $D \subseteq \mathbb{C}$, and agree on a set E which has an accumulation point $a \in D$, then $f_1 \equiv f_2$ in D.

8. Let $G = \{z \in \mathbb{C} : |z| < 2, z \neq \pm 1\}$. Find all bijective conformal maps $\Phi\colon G \to G$.

13. Fall 2013 (October)

1. The Bernoulli numbers $B_0, B_1, \ldots$ are defined by
$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}.$$
(The function on the left hand side is defined to be equal to 1 at $t = 0$.)
   (a) What is the radius of convergence of the above Taylor series? Explain.
   (b) Compute B_0 and B_1 and show that $B_{2n+1} = 0$ for all $n \geq 1$.

2. (a) Show that the series
$$\sum_{n=0}^{\infty} \left(\frac{1}{s+n} + \frac{1}{s-n} \right)$$
is pointwise convergent for every $s \in \mathbb{R} \setminus \mathbb{Z}$.
   (b) Use Weierstrass M-test (or any other method) to show that the resulting function is continuous on its domain $\mathbb{R} \setminus \mathbb{Z}$.

3. A function $f \colon X \to X$ from a metric space (X, d) to itself is called a *contraction* if there is a constant $K < 1$ such that $d(f(x), f(y)) \leq K\, d(x, y)$ for all $x, y \in X$. An element $x \in X$ is a *fixed point* if $f(x) = x$.
   (a) Show that a contractive map from a *complete* metric space to itself has a unique fixed point.
   (b) Give counterexamples to show that both conditions (completeness of X and contractive property of f) are needed in general.

4. Consider the derivative operator $T \colon C^1[0, 1] \to C[0, 1]$, $T(f) = f'$, from the space of continuously differentiable functions on the interval $[0, 1]$ to the space of continuous functions on $[0, 1]$. Show that this map is not continuous with respect to the uniform metric
$$d(f, g) = \sup\{|f(x) - g(x)| : x \in [0, 1]\}.$$

5. Show that the following limit exists:
$$\lim_{x \to \infty} \int_0^x \sin t^2 \, dt.$$
(Hint: Use the substitution $u = t^2$ as a first step.)

6. (a) Define what it means for a topological space to be *connected*, or *path connected*.
   (b) Show that a path connected topological space is connected.
   (c) Give an example of a topological space which is connected but not path connected.

7. Suppose $f(z)$ is an entire function and $|f(z)| \geq |e^z|$ for all z. Prove: $f(z) = ce^z$ for some constant c.

8. Let $D = \{z \in \mathbb{C} : |z| < 1\}$. Suppose $f \colon D \to \mathbb{C}$ is analytic and $|f(z)| \leq \frac{1}{1-|z|}$. Show $|f'(z)| \leq \frac{4}{(1-|z|)^2}$.

9. Evaluate
$$\frac{1}{2\pi i} \int_{|z|=1} \frac{z^2}{4e^z - z} dz.$$

10. Let $D = \{z \in \mathbb{C} : |z| < 1\}$. Suppose $f \colon D \to D$ is analytic and has (at least) two fixed points (i.e., there are $a, b \in D$ such that $f(a) = a$, $f(b) = b$, $a \neq b$.) Prove: $f(z) = z$.

14. Spring 2013 (May)

1. Let $\mathcal{L} \subset \mathbb{C}$ be a real line passing through the origin. Prove that if points $z_1, \ldots, z_n$ lie on one side of $\mathcal{L}$, then
$$\sum_{k=1}^{n} z_k \neq 0.$$

2. Suppose f and g are continuous functions on $[0,1]$. Prove that
$$g(x) = (1-x)\left(g(0) - \int_0^x t f(t)\,dt\right) + x\left(g(1) - \int_x^1 (1-t) f(t)\,dt\right)$$
for all $x \in [0,1]$ if and only if g is twice differentiable and $g'' = f$ on $[0,1]$.

3. Evaluate
$$\iiiint_S (w^2 + x^2 + y^2 + z^2)\,dw\,dx\,dy\,dz$$
where
$$S = \{(w,x,y,z) \in \mathbb{R}^4 : w^2 + x^2 \leq \sqrt{y^2 + z^2}\}.$$

4. Suppose $f(x)$ is a function holomorphic on the open unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ and continuous on the closed disc $\overline{\Delta}$. Suppose that $\operatorname{Im} f(z) = 0$ for $|z| = 1$. Give the best possible description of $f(x)$.

5. Let $P \subset \mathbb{R}^2$ be a polygon with non-self-intersecting boundary, having vertices $v_j = (x_j, y_j)$, for $j = 1, \ldots, n$, and edges $\overline{v_1 v_2}, \ldots, \overline{v_{n-1} v_n}$ and $\overline{v_n v_1}$. Use Green's theorem to prove that the area of P is
$$\frac{1}{2}\Big|(x_1 y_2 + x_2 y_3 + \cdots + x_{n-1} y_n + x_n y_1) - (y_1 x_2 + y_2 x_3 + \cdots + y_{n-1} x_n + y_n x_1)\Big|.$$

6. Find the general form of a linear-fractional transformation that preserves two opposite points on the Riemann sphere.

7. Suppose that $f(z)$ is a holomorphic function with an isolated singularity at a point z_0. Suppose that the singularity is not removable. Prove that the function $e^{f(z)}$ has an essential singularity at z_0.

8. Let $C(\mathbb{R}^m, \mathbb{R})$ be the set of all continuous functions from $\mathbb{R}^m$ to $\mathbb{R}$, and define
$$d(f,g) = \sum_{k=1}^{\infty} 2^{-k} \min\left\{1, \max_{|x| \leq k} |f(x) - g(x)|\right\}.$$
   (a) Prove that d is a metric on $C(\mathbb{R}^m, \mathbb{R})$.

   (b) Suppose $f, f_n \in C(\mathbb{R}^m, \mathbb{R})$ for $n \in \mathbb{N}$. Prove that $d(f_n, f) \to 0$ if and only if for every compact $K \subseteq \mathbb{R}^m$ and every $\epsilon > 0$, there exists an N such that $|f_n(x) - f(x)| < \epsilon$ whenever $x \in K$ and $n \geq N$.

15. Fall 2012 (September)

1. Solve the boundary value problem
$$xf'' = 4f' - 25x^9 f, \quad f(0) = 0,\ f(1) = 1$$
by making the substitution $x^5 = t$.

2. Given a sequence of continuous functions $\phi_n : \mathbb{R} \to [0, \infty)$ satisfying
$$\int_{\mathbb{R}} \phi_n(t)\, dt = 1 \ \text{ and } \ \lim_{n\to\infty} \int_{|t|>\delta} \phi_n(t)\, dt = 0, \ \text{for all } \delta > 0$$
show that
$$\lim_{n\to\infty} \int_{\mathbb{R}} \phi_n(x-t) f(x)\, dt$$
whenever $f : \mathbb{R} \to \mathbb{R}$ is continuous at x and bounded on $\mathbb{R}$.

3. Let $(y_n)_{n=1}^{\infty}$ be a sequence of real numbers and define $f : \mathbb{R} \to \mathbb{R}$ by setting
$$f(x) = \inf_{n\in\mathbb{Z}_+} n|x - y_n|.$$
   (a) Show that if (y_n) has no accumulation point, then f is continuous.

   (b) Find a sequence (y_n) for which f is not continuous. Justify your answer.

4. Let C be the set of continuous real-valued functions on $[0, 1]$. Given $f, g \in C$, define
$$d(f, g) = \sup_{x\in[0,1]} |f(x) - g(x)| \text{ and } \rho(f, g) = \int_0^1 |f(t) - g(t)| dt.$$
   (a) Show that d and ρ are metrics on C.

   (b) Prove that (C, d) is complete.

   (c) Show that (C, ρ) is not complete.

5. Let γ be the circle with radius 2 centered at 1 traversed one time counterclockwise. Evaluate the integrals:
   (a) $\int_\gamma \frac{e^{2z}}{(1+z^2)^2} dz$

   (b) $\int_\gamma \frac{sin(\pi z)}{z^2-2z} dz$.

6. How many solutions, counted with multiplicities, does the equation $e^{-z} = 2z^3 + 3z + 1$ have in the disc $|z| < 2$?

7. Evaluate $\int_0^\infty \frac{dx}{1+x^4}$.

8. Find three different Laurent series for $f(z) = \frac{1}{z-2} - \frac{1}{z} + \frac{1}{(z+1)^2}$ about $z_0 = 1$ and state their regions of convergence.

16. Spring 2012 (May)

1. Let (A, α), (B, β) and (C, γ) be metric spaces, and define
$$d((a_1, b_1), (a_2, b_2)) = \sqrt{\alpha(a_1, a_2)^2 + \beta(b_1, b_2)^2}.$$
   (a) Prove that $(A \times B, d)$ is a metric space.

   (b) Suppose K is a compact subset of B, V is an open subset of C, and f is a continuous map from $(A \times B, d)$ to (C, γ). Prove that
$$W = \{a \in A : f(a, b) \in V \text{ for all } b \in K\}$$
   is an open subset of A.

2. Solve the initial value problem,
$$f'''(x) - 3f''(x) + 4f(x) = 4x + 4, \quad f(0) = 2,\ f'(0) = 4,\ f''(0) = 8.$$

3. Let $R = \{(u, v) \in \mathbb{R}^2 : u > 0,\ u^2 v > 1\}$.

   (a) Find the image of R under the map $(u, v) \mapsto (u^{-2}v^{-1}, u^{-1}v^{-2})$.

   (b) Evaluate
$$\iint_R \frac{1}{u^4v^4 + u^2}\, du\, dv.$$
   Justify your answer.

4. Suppose f is a non-negative, decreasing function on $(0, \infty)$ such that $\int_0^\infty f(x)dx < \infty$ and let $g(x) = \sum_{n=1}^\infty f(2^n x)$ for each $x > 0$. Prove that $g(x) < \infty$ for all $x > 0$, that the sum converges uniformly to g on any compact subinterval of $(0, \infty)$, and that
$$\int_0^\infty g(x)\, dx = \int_0^\infty f(x)\, dx.$$

5. Evaluate
$$\int_{-\infty}^\infty \frac{x^2}{1 + x^4}\, dx.$$

6. Find three different Laurent series for
$$f(z) = \frac{1}{z} + \frac{1}{(z+1)^2} + \frac{1}{z - 2}$$
   about $z_0 = 0$ and state their regions of convergence.

7. Let γ be the limaçon $r = \frac{3}{2} + 3\cos\theta$ traversed one time counterclockwise. Evaluate the integrals

   (a) $\int_\gamma \frac{e^{2z}}{(1+z^2)^2}\, dz$

   (b) $\int_\gamma \frac{\sin(\pi z)}{z^2 - 3z + 2}\, dz$.

8. Let Ω be a non-empty open subset of $\mathbb{C}$, let $(f_n)_{n=1}^\infty$ be a sequence of functions holomorphic on Ω, and let $f : \Omega \to \mathbb{C}$ be a non-constant function. Suppose that $f_n \to f$ as $n \to \infty$ uniformly on every compact subset of Ω. Prove that, if $p \in \Omega$ and $f(p) = 0$, then for every open neighbourhood U of p in Ω there exists $N \in \mathbb{N}$ such that f_n has a zero in U for all $n \geq N$.

9. BONUS: Prove that there is no function f analytic in the disc $D = \{z \in \mathbb{C} : |z| < 2012\}$ and such that $|f(z)| \to \infty$ as $|z| \to 2012^-$.

17. Fall 2011 (September)

1. Prove Schwarz's Theorem: If $f(z)$ is analytic for $|z| \le R$ and if $f(0) = 0$ and $|f(z)| \le M$, then
$$|f(z)| \le \frac{M|z|}{R}.$$

2. Prove that
$$\int_0^{2\pi} (\cos\theta)^{2n}\, d\theta = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} 2\pi.$$

3. Prove that all the roots of $p(z) = z^7 - 5z^3 + 12$ lie between the circles $|z| = 1$ and $|z| = 2$.

4. Show that the function $f\colon \{z \in \mathbb{C} : |z| > 1\} \to \mathbb{C}$ defined by $f(z) = \frac{1}{2}\left(z + \frac{1}{z}\right)$ is injective, and find its image.

5. Let f be a holomorphic function on the open unit disc. Suppose that there exists an open set R on the unit circle with the property that $\lim f(z) = 1$, as z approaches R (z is in the disc). Prove that f is identically 1.

6. (a) Consider the series $\sum_{n=1}^{\infty} \alpha_n \beta_n$ where $\alpha_n, \beta_n \in \mathbb{R}$. Prove that if $a_1, a_2, \ldots$ is a non-increasing sequence and the partial sums $B_N = \sum_{n=1}^{N} \beta_n$ are uniformly bounded in absolute value by some $L > 0$ (i.e., $|B_N| \le L$ for $N = 1, 2, \ldots$), then
$$S_N = \left|\sum_{n=1}^{N} \alpha_n \beta_n\right| \le L(|\alpha_1| + 2|\alpha_N|)$$
(Hint: transform the formula for S_N so that the β_n are replaced by B_n.)

   (b) Use (6a) to prove the Dirichlet test for convergence of series: the sum
$$\sum_{n=1}^{\infty} a_n b_n$$
converges whenever the sequence $B_N = \sum_{n=1}^{N} b_n$ is bounded and $\{a_n\}$ is a decreasing sequence such that $\lim_{n\to\infty} a_n = 0$.

   (c) Use the Dirichlet test for convergence to show that the series
$$\sum_{n=1}^{\infty} \frac{1}{n} \sin(n)$$
converges.

   (Hint: use $2\sin(u)\sin(v) = \cos(u - v) - \cos(u + v)$).

7. Let $f(x)$ be a differentiable function of one real variable defined for $x > 0$. Suppose that, for all $x > 0$,
$$|f(x)| \le \frac{C}{x^k}$$
where $C > 0$ and $k \ge 0$. Further, assume that k is the best possible rate of growth for f, i.e., the inequality does not hold for any smaller values of k (and any $C > 0$). Finally, suppose that the same estimate holds for $|f'(x)|$, possibly with a different constant C, but the same k. Prove that
$$\lim_{x \to 0^+} f(x)$$
exists.

8. Let X be the metric space of continuous functions on the interval $[0,1]$ with the metric defined as

$$d(f,g) = \max_{0 \leq x \leq 1} |f(x) - g(x)|$$

and let Y be the metric space defined on the same collection of functions but with the metric

$$\rho(f,g) = \left(\int_0^1 |f(x) - g(x)|^2 \, dx \right)^{1/2}.$$

Show that X is a complete metric space, while Y is a metric space which is not complete.

18. Spring 2011 (May)

1. Let
$$T_n = \left\{ (x_1, y_1, x_2, y_2, \dots, x_n, y_n) \in [0,\infty)^{2n} \colon \sum_{k=1}^{n} \max(x_n, y_n)^2 \le 1 \right\}$$
and let α_n be the $2n$-dimensional volume of T_n. State and prove a simple formula for α_n in terms of n.

2. Let h be a real-valued, continuous, non-negative, non-increasing function on $[0,\infty)$. For $n = 1, 2, \dots$, define h_n by
$$h_n(x) = n \int_x^{x+1/n} h(t)dt.$$
Prove that $h_1, h_2, \dots$ is a non-decreasing sequence of differentiable, non-negative, non-increasing functions that converges pointwise to h.

3. Let X be a bounded subset of $\mathbb{R}^n$ and $f \colon X \to \mathbb{R}$ be a continuous function. Suppose that, for each $y \in \mathbb{R}^n \setminus X$, there exists a $\delta_y > 0$ and a constant B_y such that $|f(x)| \le B_y$ for all $x \in X$ such that $|x - y| < \delta_y$. Prove that f is bounded.

4. Let (T, d) and (Y, ρ) be complete metric spaces. Suppose S is a subset of T, $f \colon S \to Y$ is uniformly continuous, and t is in the closure of S but not in S. Prove that f extends to a continuous function $g \colon S \cup \{t\} \to Y$. That is, prove that there exists $y \in Y$ such that
$$g(s) = \begin{cases} f(s) & s \in S \\ y & s = t \end{cases}$$
is continuous at t.

5. Expand $\frac{1}{(z-1)(z-2)}$ in a Laurent series centered at $z = 0$ and converging in the annulus $0 < |z| < 2$.

6. Evaluate
$$\int_0^{2\pi} \frac{\sin^2\theta}{5 + 4\cos\theta}\, d\theta.$$

7. Fix $n \le 1$, $r > 0$, and $\lambda = \rho e^{i\Phi}$. What is the maximum modulus of $z + \lambda^n$ over the disc $|z| \le r$? Where does $z + \lambda^n$ attain its maximum modulus over the disc?

8. Show that the image of a non-constant entire function is dense in $\mathbb{C}$.

19. Fall 2010 (October)

1. Find a Möbius transformation mapping the half-plane $\{z : \operatorname{Re} z < 1\}$ onto $\{z : |z-1| > 2\}$.

2. Suppose $g : [0,1] \to \mathbb{C}$ is continuous. Prove that g is uniformly continuous.

3. Classify the singularity of $\frac{z^2(z-1)}{(1-cosz)log(1+z)}$ at $z = 0$.

4. Let (X, d) be a metric space and a be a point of X. Define

$$\rho(x,y) = \begin{cases} d(x,a) + d(a,y) & x \neq y \\ 0 & x = y. \end{cases}$$

   (a) Show that (X, ρ) is a metric space.

   (b) Show that if a subset of X is open in (X, d) then it is open in (X, ρ).

   (c) Give an example of a metric space (X, d) and a point $a \in X$ such that the topologies on (X, d) and (X, ρ) coincide but are not just the discrete topology.

5. Let f be an even meromorphic function, that is to say, let f be a meromorphic function such that $f(-z) = f(z)$ for all z, and suppose that f has a pole at 0. Show that the residue of f at 0 is equal to 0.

6. [duplicate of #7 of October 1996]

7. Evaluate

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^3} dx.$$

8. In a game of hide and seek on the complex plane the hider is hiding in a tree at the origin. The seeker runs counterclockwise along the unit circle from 1 to -1 at unit speed. When the seeker reaches $e^{i\pi/4}$, the hider leaves the tree and runs at a constant speed to 1 always keeping the tree directly between himself and the seeker. The hider arrives at 1 at the same time that the seeker arrives at -1. What path does the hider follow?

   (Hint: Express the position of the hider in polar form, find the argument, and use constant speed to determine the modulus.)

9. Apply the maximum principle to find the smallest number A for which the inequality $|\sin z| \leq A|z|$ is satisfied in $\{z : |z| \leq 1\}$.

20. Spring 2010 (May)

1. Suppose that f is holomorphic for $|z| < 1$. Suppose that $|f(z)| \leq 1$ for all $|z| < 1$, and
$$f(0) = f'(0) = \cdots = f^{(k-1)}(0) = 0.$$
Prove that $|f(z)| \leq |z|^k$ for all $|z| < 1$.

2. Let γ be the positively oriented circle $|z| = 1/2$. Evaluate
$$\int_\gamma \frac{e^{1/z}}{1-z} dz.$$

3. Consider the linear fractional transformation $f(z) = \frac{a+b}{cz+d}$ with $ad - bc \neq 0$ as a map on the extended complex plane $\mathbb{C} \cup \{\infty\}$, i.e., on the Riemann sphere. Show that f maps circles in the extended complex plane to circles.

   Hint: First prove that f can be written as $f(z) = A + \frac{B}{z+C}$.

4. Let $P(z)$ be a complex polynomial in z. Suppose that all zeros of $P(z)$ are contained in the upper half plane. Prove that the zeros of $P'(z)$ are also contained in the upper half plane.

   Hint: Consider $\frac{P'}{P}$ (logarithmic differentiation).

5. Suppose
$$f(z) = az^2 + bz\bar{z} + c\bar{z}^2$$
where a, b, and c are fixed complex numbers.

   (a) Show that $f(z)$ is complex differentiable at z if and only if $bz + 2c\bar{z} = 0$.

   (b) Where is $f(z)$ analytic?

   Justify your answers.

6. (a) Show that the area of a planar region delimited by a closed simple curve C is given by $\frac{1}{2}\int_C x\,dy - y\,dx$.

   (b) Compute $\int_C (2xy - x^2)dx + (x + y^2)dy$, where C is the boundary of the bounded region delimited by the graphs of $y = x^2$ and $y^2 = x$.

7. (a) Show that every subspace of a separable metric space is separable.

   (b) Let X be a separable metric space and let $Y \subset X$ be any subspace. Given $N \in \mathbb{N}$, construct a sequence $\{a_k\}$ where
$$a_k = (a_{k,1}, a_{k,2}, \ldots, a_{k,N}) \in Y^N,$$
   with the property that, given any $y \in Y^N$, there is a subsequence $\{a_{k_i}\}$ converging to y.

8. Let (X, d) be a complete metric space. Show that a contraction $f \colon X \to \mathbb{R}$ is necessarily continuous and has precisely one fixed point. Recall that f is a contraction iff there is a constant $0 < C < 1$ such that
$$d(f(x), f(y)) \leq C\, d(x, y) \quad \text{for all} \quad x, y \in X.$$

9. Let $X = C[0, 1]$ with the topology of uniform convergence.

   (a) Is the subspace $\mathcal{P}$ of polynomials open in X?

   (b) Is $\mathcal{P}$ closed?

   Justify your answers.

10. *Helly's selection principle* states that given a sequence (f_n) of nondecreasing functions $f_n\colon [0,1] \to [a,b]$, there exists a subsequence (f_{n_k}) and a function $F\colon [0,1] \to [a,b]$ such that

$$\lim_{k\to\infty} f_{n_k}(x) = F(x) \text{ for any } x \in [0,1]$$

The proof is divided into three steps.

(a) Show that we can find a subsequence (f_{n_k}) that converges to a nondecreasing function G defined on all rational points $\{r_1, r_2, r_3, \ldots\}$ of $[0,1]$.

(b) Define $H\colon [0,1] \to [a,b]$ by setting

$$H(x) = \lim_{\substack{r\to x^- \\ r\in\mathbb{Q}\cap[0,1]}} G(r).$$

Show that H is the limit of (f_{n_k}) at each continuity point of H.

(c) Now, recall that a nondecreasing real valued function of a real variable has at most countably many discontinuous points. Use a diagonal argument to find a subsequence of (f_{n_k}) that converges everywhere on $[0,1]$ to some function F.

21. Fall 2009 (October)

1. Let $\mathbb{R}$ be the set of real numbers with the usual metric and let $\mathbb{R}_1$ be the set $\mathbb{R}$ with the distance function $\rho(x, y) = |\tan^{-1}(x) - \tan^{-1}(y)|$.
   (a) Prove that ρ is a metric.
   (b) Prove that the identity map $\mathbb{R} \to \mathbb{R}_1$ is a homeomorphism.
   (c) Prove that $\mathbb{R}_1$ is not complete.
   (d) Define a metric on the set $X = \{1/n : n = 1, 2, \ldots\}$ for which X is complete.

2. A function $f : \mathbb{R} \to \mathbb{R}$ is called convex if and only if
$$f(ax + (1-a)y) \leq af(x) + (1-a)f(y)$$
for $x, y \in \mathbb{R}$ and $a \in [0, 1]$. Prove that a convex function is continuous on $\mathbb{R}$.

3. Evaluate
$$\int_0^1 \int_y^1 \sin(x^2)\, dx\, dy.$$

4. Find a Taylor series and two Laurent series for
$$f(z) = \frac{1}{z} + \frac{1}{z-3}$$
about $z = 1$, and state the region where each converges.

5. Let G be a bounded, open, connected subset of $\mathbb{C}$. Suppose that f is continuous on $\bar{G}$ and analytic on G and that there is a $c > 0$ such that $|f(z)| = c$ for all $z \in \partial G = \bar{G} \setminus G$. Prove that f is constant on G or else f has a zero in G.

6. [duplicate of #2 on Fall 1998]

7. Classify (as removable, essential, or pole) the singularities of
$$f(z) = \csc(z) - \frac{1}{z}$$
in the Riemann sphere $\mathbb{C} \cup \{\infty\}$.

8. Let X be compact and let $\{f_n\}$ be a sequence of functions from X into $\mathbb{R}$. Suppose that f is a continuous function on X such that $\lim_{n\to\infty} f_n(x) = f(x)$ for each $x \in X$. If $f_n(x) \leq f_{n+1}(x)$ for all n and all $x \in X$, prove that $f_n \to f$ uniformly on X.

9. Evaluate
$$\int_0^\infty \frac{dx}{x^5 + 1}.$$

22. Fall 2008 (October)

1. Suppose f is a non-constant entire function. Show that $f(\mathbb{C})$ is dense set in $\mathbb{C}$.

2. Suppose X and Y are metric spaces and $f : X \to Y$ is continuous. Show that if X is compact and f is onto, then Y is complete.

3. A function $f : \mathbb{R} \to \mathbb{R}$ is called Lipschitz with constant $C \in [0, \infty]$ provided
$$|f(x) - f(y)| \leq C|x - y|$$
for all $x, y \in \mathbb{R}$. Suppose that f_n is Lipschitz with constant C_n, for $n = 1, 2, \ldots$, and suppose that $\lim_{n\to\infty} f_n(x) = f(x)$ for each $x \in \mathbb{R}$. Prove that f is Lipschitz with constant
$$C = \limsup_{n\to\infty} C_n.$$

4. Compute
$$\int_{|z|=3} \frac{e^{\frac{1}{z-1}}}{z-2} dz.$$

5. Find the zeros of the function
$$f(z) = z^7 - 3z^5 - 12z^4 + z^2 + z + 1$$
in $D = \{z \in \mathbb{C} \colon |z| < 1\}$.

6. Show that
$$\int_0^\infty \frac{1}{1 + x^3}\, dx = \frac{2\pi}{3\sqrt{3}}.$$
Hint: Set $\omega = e^{\frac{i\pi}{3}}$, let M be a positive real number, and let γ_M be the closed, positively oriented contour consisting of the two segments $[M\omega^2, 0]$ and $[0, M]$ and the circular arc from M to $M\omega^2$, centered at 0. Use the residue theorem to compute the integral of $1/(1 + z^3)$ over the contour γ_M.

7. Let R be the region in the first quadrant bounded by the curve $x^3 + y^3 = 1$. Make the change of variables $u = x^3 + y^3$, $v = y/x$ to evaluate
$$\iint_R x\, dx\, dy.$$
You may use the result of question 6.

8. Either give an example of a function that is analytic on $D = \{z \in \mathbb{C} \colon |z| < 1\}$ and satisfies $f(0) = 0$, $f(1/4) = i$, and $|f(z)| \leq 2$ for all $z \in D$ or prove that such a function does not exist.

9. Show that the boundary value problem $y' = y^2 + x$, $y(0) = 0$, has no continuously differentiable solution that is valid on the interval $[0, \infty)$.

   Hint: Show that $y(4 - 2^{-n}) \geq 2^{n+2}$ by induction.

23. Fall 2007 (October)

1. Let $f : X \to Y$ be a map between metric spaces.

   (a) Prove that f maps closed sets onto closed sets if and only if $f(\overline{A}) \supset \overline{f(A)}$ for all $A \subset X$.

   (b) If f is continuous, prove that $f(\overline{A}) \subset \overline{f(A)}$ for all $A \subset X$.

2. (a) For a subset A of a metric space, define the terms boundary of A and limit point of A.

   (b) Prove that the boundary ∂A and the set of limit points A' are closed.

   (c) Can ∂A be a non-empty open set? Prove your answer.

3. Find the number of zeroes (counting multiplicities) of $f(z) = 3e^z - z^{2007}$ in $1 \leq |z| \leq 2$.

4. Find all solutions of the equation $\cos(2z) = 5$.

5. Evaluate

$$\int_{|z|=2} \frac{e^{z+i}}{z^3 + z^2} dz.$$

6. Let $S = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle. Let $f(z) = \overline{z}$. Prove that $f|_S$ cannot be approximated by holomorphic polynomials (i.e. polynomials in z) uniformly on S.

7. Prove that all the roots of the Legendre polynomials

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2 - 1)^n$$

   belong to the interval $(-1, 1)$ for $n = 1, 2, 3, \cdots$.

   Hint: Find inductively the roots of the polynomials $\frac{d^k}{dx^k}(x^2 - 1)^n, k = 1, 2, \cdots, n$ counting multiplicities.

8. A function f defined on a subset E of $\mathbb{R}$ is said to be upper semi-continuous (u.s.c.) if

   (a) $-\infty \leq f(x) \infty$

   (b) For each $\alpha \in \mathbb{R}$, the set $\{x \in \mathbb{R} : f(x) < \alpha\}$ is open.

   Note that f is allowed to attain the value $-\infty$.

9. A function f is said to be lower semi-continuous (l.s.c.) if $-f$ is upper semi-continuous.

   (a) Prove that a function $f : \mathbb{R} \to (-\infty, \infty)$ is continuous if and only if it is both u.s.c. and l.s.c.

   (b) Prove that an u.s.c. function $f : [0, 1] \to [-\infty, \infty)$ attains its maximum on $[0, 1]$.

24. Spring 2007 (June)

1. Let γ be the circle $\{z \in \mathbb{C} : |z - 3| = 2\}$, traversed once in the positive direction. Evaluate,
$$\int_\gamma \frac{z-3}{\sin(z)}\, dz.$$

2. Let R be the region in the first quadrant bounded by the circle $x^2 + y^2 = 1$ and the lines $y = 0$ and $y = x$. Evaluate
$$\iint_R xy + \frac{y^3}{x}\, dx\, dy.$$

3. Find the complex number w satisfying $e^w = -1 + i$ such that $|w + 3i|$ is as small as possible.

4. Prove or find a counterexample: If X is a non-empty set and d_1 and d_2 are metrics on X, then d defined by $d(x, y) = \min(d_1(x, y), d_2(x, y))$ is also a metric on X.

5. Suppose $w \in \mathbb{C}$, $R > 0$, f is analytic in $\{z \in \mathbb{C} : 0 < |z - w| < R\}$ and f has a pole at w. Prove that for some $r > 0$, the function f/f' is analytic in $\{z \in \mathbb{C} : 0 < |z - w| < r\}$ and has a simple pole at w.

6. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a real analytic function. Further, assume that there exists a sequence of real numbers $\{a_\nu\}_{\nu=1}^{\infty}$, such that $a_\nu \to 0$ as $\nu \to \infty$ and $f(a_\nu) = 0$. Prove that $f \equiv 0$.

7. Let $f_n(x)$ be continuous functions on $[0, 1]$, for $n = 1, 2, \ldots$, that converge pointwise to a function $f(x)$. Suppose that, for any $\varepsilon > 0$ and any $N > 0$, there exists at least one $n' > N$ independent of x such that $|f_{n'}(x) - f(x)| < \varepsilon$. Prove that $f(x)$ is continuous on $[0, 1]$.

25. Fall 2006 (October)

1. Given an example of a function on the interval $[-1, 1]$ which is differentiable at the origin, but is not differentiable on any open interval containing the origin.

2. Use Green's theorem to evaluate $\int_\gamma P\,dx + Q\,dy$ where
$$P(x, y) = \cos(x), \quad Q(x, y) = 3x + 4y + 1,$$
and γ is the circle $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 4\}$ traversed in the clockwise direction.

3. Suppose f is analytic on the disc $\{z \in \mathbb{C} : |z| \leq 1\}$. Further, suppose that
   - $|f(z)| \leq 2$ for $|z| = 1$ and $\mathrm{Re}(z) \geq 0$;
   - $|f(z)| \leq 18$ for $|z| = 1$ and $\mathrm{Re}(z) \leq 0$.

   Prove that $|f(0)| \leq 6$.

4. Let f and g be two linearly independent entire functions. Prove that there is a sequence $\{z_n\}$ of complex numbers such that $|f(z_n)| \geq n|g(z_n)|$ for every positive integer n.

5. Compute $\int_{|z|=2} \frac{e^{2z}}{z^2(z-i)(z+5)}\,dz$.

6. Let X be the set of all continuous functions $[0, 1] \to \mathbb{R}$ with the metric defined by
$$\rho(f, g) = \sup_{0 \leq x \leq 1} |f(x) - g(x)| \quad \text{for all } f, g \in X.$$
Prove that (X, ρ) is complete (you do not need to prove that ρ is a metric on X).

7. Let $\rho_1 \colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be the function defined by
$$\rho_1(x, y) = |\arctan(x) - \arctan(y)|$$
for all $x, y \in \mathbb{R}$.
   (a) Prove that ρ_1 is a metric on $\mathbb{R}$.

   (b) Prove that $(\mathbb{R}, \rho_1)$ is not complete.

   (c) Let ρ_2 denote the usual metric on $\mathbb{R}$; that is, $\rho_2(x, y) = |x - y|$ for all $x, y \in \mathbb{R}$. Prove that the identity function $(\mathbb{R}, \rho_1) \to (\mathbb{R}, \rho_2)$ is a homeomorphism.

8. Let $X = \{\frac{1}{n} : n \in \mathbb{N}\}$. Define a metric ρ on X for which (X, ρ) is complete.

9. Suppose that $f \colon [0, 1] \to \mathbb{R}$ is a continuous function which is right differentiable on $[0, 1)$; that is, for each $x \in [0, 1)$, the limit
$$Rf(x) := \lim_{h \to 0^+} \frac{f(x + h) - f(x)}{h}$$
exists. Prove that if $Rf(x) > 0$ for all $x \in [0, 1)$, then f is a strictly increasing function.

10. The classical Weierstrass theorem claims that any continuous function $[0, 1] \to \mathbb{R}$ can be uniformly approximated by a sequence of polynomials. Assuming the Weierstrass theorem, prove that in fact, given a positive integer $k > 0$, one only needs polynomials that are linear combinations of the terms $1, x^k, x^{2k}, x^{3k}, \ldots$ for such an approximation.

26. Spring 2006 (April)

1. (a) Let $\mathcal{P}(\mathbb{N})$ be the power set of $\mathbb{N}$, i.e. the set of all subsets of $\mathbb{N}$. Prove that $\mathcal{P}(\mathbb{N})$ is not countable.

   (b) Let E be any set. Prove that there is no surjective map $f\colon E \to \mathcal{P}(E)$.

2. Suppose f is an entire function, $|f(z)| \leq 1$ for $|z| < 1$, $f(0) = 0$, $f'(0) = 0$. Show that $|f''(0)| \leq 2$.

3. Suppose f is analytic in $\{z \in \mathbb{C}\colon |z| < 1\}$ and it has a zero of order $k \geq 2$ at $z = 0$. What type of singularity does the function $\frac{f''(z)}{f(z)}$ have at $z = 0$? If is isolated, then determine the residue.

4. (a) Suppose X is a complete metric space such that there exists a positive constant M and $\|x - y\| < M$ for all $x, y \in X$. Prove or disprove: X is compact.

   (b) Suppose X is a complete metric subspace of a compact metric space Y. Prove or disprove: X is compact.

5. Find the number of zeros of $p(z) = 6z^4 + z^3 - 2z^2 + z - 1$ in the disc $|z| \leq 1$.

6. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous non-constant periodic function, i.e. $f(x+c) = f(x)$ for all x and some constant $c \neq 0$. Such c is called a *period* of f. Prove that there exists $p > 0$ such that p is a period but any number c with $0 < c < p$ is not a period.

7. Suppose that f is an entire function and $f(z) = f(z+2) = f(z+i)$ for all $z \in \mathbb{C}$. Prove that f is constant.

8. Evaluate
$$\int_{|z|=2} \frac{\cos z}{(z-4)(z+i)^3} dz.$$

9. (a) Give an example of a sequence of continuous functions $f_n(x)$ on the interval $[0,1]$ which converges pointwise to a function $f(x)$ that is not continuous on $[0,1]$.

   (b) Give an example of a sequence of Riemann integrable functions $f_n(x)$ on the interval $[0,1]$ which converges pointwise to a function $f(x)$ that is not Riemann integrable on $[0,1]$.

27. Spring 2005 (May)

1. Let C_r be the circle in the complex plane with radius $r > 0$ and centre 0 once in the counterclockwise direction. Calculate
$$\int_{C_r} \frac{z^2}{z - \sin(z)} dz$$
for $r > 0$ sufficiently small.

2. Let S be the region in the (u, v)-plane consisting of those points inside the square with vertices $(2, 0)$, $(3, 1)$, $(2, 2)$, and $(1, 1)$. Evaluate
$$\iint_S \sqrt{u^2 - v^2}\, du\, dv$$
by making the substitution $u = x + y$, $v = x - y$.

3. Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic with $f(0) = 1$ and
$$\left| \frac{f(z)}{1 - z^2} \right| \leq 2, \text{ for } z \in \mathbb{C} \setminus \{1, -1\}.$$
Show that $f(z) = 1 - z^2$ for all $z \in \mathbb{C}$.

4. Let (X, p) and (Y, q) be metric spaces and $Z = X \times Y$. Define $r : Z \times Z \to [0, \infty)$ by
$$r((x_1, y_1), (x_2, y_2)) = \begin{cases} p(x_1, x_2), & x_1 \neq x_2 \\ q(y_1, y_2), & x_1 = x_2. \end{cases}$$
Prove that r is a metric on Z if and only if
$$\sup_{y_1, y_2} q(y_1, y_2) \leq 2 \inf_{x_1 \neq x_2} p(x_1, x_2).$$

5. Let S be a circle of positive radius and L be a line in the complex plane. Show that if f is an entire function and $f(S) \subset L$, then f is constant.

6. Consider the boundary value problem
$$xy'' = 4y' - 25x^9 y, \quad y(0) = 0, \quad y(1) = 1,$$
where y is a function of x. Solve the problem by making the substitution $t = x^5$.

7. Show that the function $f : \{z \in \mathbb{C} : |z| > 1\} \to \mathbb{C}$, defined by
$$f(z) = \frac{1}{2}\left(z + \frac{1}{z}\right),$$
is injective and find its image.

8. Suppose $\{f_n\}$ is a sequence of real-valued functions that converge to 0 pointwise on $[0, 1]$. Prove that if each function f_n is non-increasing, then the convergence is uniform.

28. Fall 2004 (October)

1. Evaluate the (triple) integral
$$\iiint_S x^2 + y^2 + z^2 \, dx \, dy \, dz$$
over the cylinder
$$S = \{(x, y, z) \in \mathbb{R}^3 \colon x^2 + y^2 \leq 1 \text{ and } 0 \leq z \leq 3\}.$$

2. Let $g_n \colon [0, 1] \to [0, 1]$ be continuous functions for $n = 1, 2, \ldots$ satisfying $\lim_{n\to\infty} g_n(x) = 0$ for each $x \in [0, 1]$. Suppose for each n that $y = f_n(x)$ is a solution to the boundary value problem
$$y' + 2xy = g_n(x), \; y(0) = 0.$$
Prove that $\lim_{n\to\infty} f_n(x) = 0$ for each $x \in [0, 1]$.

3. (a) Verify that the function $u(x, y) = \cos(x)\cosh(y)$ is harmonic.

   (b) Find a harmonic conjugate v for u, that is, a function v such that $u+iv$ is a holomorphic function of $z = x + iy$ with $v(0, 0) = 0$.

4. Let C be a circle in the complex plane having radius 3 and center 0, traced once in the counterclockwise direction. Calculate
$$\int_C \frac{\sin(z)}{2z^4} \, dz.$$

5. Let (X, d) be a metric space and $f \colon X \to X$. Suppose that whenever $E \subset X$ and x is in the boundary of E, $f(x)$ is in the boundary of $f(E)$. Prove that f is continuous on X.

6. How many solutions, counted with multiplicities, does the equation $e^z = 2z^3 + 3z - 1$ have in the open disc $|z| < 2$? (You may assume $e < 3$).

7. Find a sequence of functions $f_n \colon [0, 1] \to \mathbb{R}$ for $n = 1, 2, \ldots$ such that $\int_0^1 f_n(x) \, dx = 1$ for each n and $\lim_{n\to\infty} f_n(x) = +\infty$ for all $x \in [0, 1]$.

8. (a) Suppose that f is an entire function, $a, b \in \mathbb{C}$ are distinct, and $|a|, |b| < R$. Show that
$$\oint_{|z|=R} \frac{f(z)}{(z-a)(z-b)} \, dz = 2\pi i \frac{f(a) - f(b)}{a - b}.$$

   (b) Use part a to prove Liouville's Theorem: if f is entire and bounded, then f is constant.

29. Spring 2004 (February)

1. (a) Show that

$$f(z) = \begin{cases} \frac{sin(z)}{z-\pi} + 1 & \text{if } z \neq \pi \\ 0 & \text{if } z = \pi \end{cases}$$

   is an entire function.

   (b) What is the order of the zero of f at $z = \pi$?

2. Show that if f is analytic on a domain D and if $|f|$ is constant, then f is constant.

3. [duplicate of #5 of Jan. 1999]

4. Expand $f(z) = \frac{1}{(z+1)(z+3)}$ in a Laurent series valid for:

   (a) $1 < |z| < 3$

   (b) $|z| > 3$

5. Find all values of $\log(-\frac{1}{2} + \frac{\sqrt{3}}{2}i)$.

6. Suppose f is analytic in the open disc $|z| < 2$, $|f|$ is bounded there by 10, and $f(1) = 0$. Find the best possible upper bound for $|f(\frac{1}{2})|$.

7. Show that $\int_0^{2\pi} \frac{d\theta}{a+b\sin(\theta)} = \frac{2\pi}{\sqrt{a^2-b^2}}$ if $a > |b|$.

30. Fall 2003 (November)

1. [duplicate of #1 on Oct. 1998]

2. [duplicate of #2 on Oct. 1998]

3. [duplicate of #3 on Oct. 1998]

4. [duplicate of #4 on Oct. 1998]

5. [duplicate of #7 on Oct. 1996]

6. Find all values of i^i.

7. Let G be a connected open subset of $\mathbb{C}$ and $f\colon G \to \mathbb{C}$ be a holomorphic map such that $f(z)$ is real for all $z \in G$. Show that f is constant.

8. Let f and g be analytic on a bounded open connected set $\Omega \subset \mathbb{C}$ and continuous on the closure $\bar{\Omega}$. Suppose g is nowhere zero in $\bar{\Omega}$. Show that if $|f| \leq |g|$ on the boundary of Ω, then $|f| \leq |g|$ on Ω.

9. Suppose that f is an entire function satisfying $|f(z)| \leq A + B|z|^k$ for all $z \in \mathbb{C}$, where A and B are constant. Let $f(z) = \sum_{j=0}^{\infty} c_j z^j$ be its power series expansion about zero. Show that all the coefficients c_j, $j > k$, are equal to zero.

10. If $|\alpha| < 1$, show that $f(z) = \frac{z-\alpha}{1-\bar{\alpha}z}$ is a one-to-one analytic function of the disc $\{z : |z| < 1\}$ onto itself. (Hint: Show $|f(z)| = 1$ when $|z| = 1$.)

11. Let $f(z) = \frac{1}{z} + \frac{1}{(z-1)^2} + \frac{1}{z+2}$. Obtain all Laurent series expansions of f about $z = 0$ and indicate where each is valid.

12. Let C be the circle $|z| = 1$ traced once in the clockwise direction. Evaluate
$$\int_C e^{\sin(1/z)} dz.$$

31. Fall 2002 (October)

1. Suppose that functions $f_n \colon [0,1] \to \mathbb{C}$ are given such that, $\forall \epsilon > 0$, $\exists \delta > 0$ such that for $n = 0, 1, 2, \ldots$, one has
$$|f_n(x) - f_n(y)| < \epsilon \text{ whenever } |x - y| < \delta.$$
Show that if f_n converges pointwise to f_0, then f_n converges uniformly to f_0.

2. Compute the integral
$$\int_{-\infty}^{\infty} \frac{dx}{x^2 + 2x + 2}.$$

3. Let $y_1, y_2, \ldots$ be a sequence of real numbers and define $f \colon \mathbb{R} \to \mathbb{R}$ by
$$f(x) = \inf_{k=1,2,\ldots} k|x - y_k|.$$
   (a) Show that if the set $\{y_1, y_2, \ldots\}$ has no accumulation point, then f is continuous.

   (b) Find a sequence $y_1, y_2, \ldots$ such that f is not continuous.

4. Suppose f is a holomorphic function on a neighbourhood of the closed unit disc such that $|f| \geq 2$ on the unit circle and $f(0) = 1$. Show that f has a zero in the unit disc.

5. Let T be the interior of the triangle with the vertices $(0,0)$, $(0,1)$, and $(1,0)$. For $(u,v) \in T$, define x and y by $x = v - uv$ and $y = 1 - u - uv$.

   (a) Show that the map $(u,v) \mapsto (x,y)$ is one-to-one and takes T onto itself.

   (b) Evaluate the integral below by making the change of variable $(x,y) \mapsto (u,v)$.
$$\iint_T \frac{dx\,dy}{\sqrt{4x + y^2}}$$

6. Find all holomorphic functions f from the unit disc $\{z \in \mathbb{C} : |z| < 1\}$ to itself such that $f(1/2) = 0$ and $f'(1/2) = 4/3$.

7. Let Ω be a connected open subset of $\mathbb{C}$ and $f \colon \Omega \to \Omega$ be a holomorphic map such that $f \circ f = f$. Show that either f is the identity map on Ω or f is constant.

8. Let $\mathcal{F} = \{y = a/x : a > 0\}$ be a family of curves in the first quadrant. Find an infinite family of curves $\mathcal{G}$ such that each $g \in \mathcal{G}$ intersects each $f \in \mathcal{F}$ at an angle of $\pi/4$.

32. Spring 1999 (January)

1. Let $X = \mathcal{C}[0,1]$ be the space of continuous $\mathbb{R}$-valued functions on $[0,1]$ with the topology of uniform convergence. Prove that the set of polynomials in X is not open.

2. Let $X = \mathbb{R}^2$ with the usual topology. Prove or give a counterexample.
   (a) The interior of the complement of a set in X is always equal to the complement of the closure of the set.
   (b) If $f\colon X \to \mathbb{R}$ is uniformly continuous and $E \subset X$ is bounded, then f is bounded on E.
   (c) Every infinite subset of X has a limit point in X.

3. Let X and Y be topological spaces with X compact. Should X, Y be Hausdorff too?!
   (a) If $f\colon X \to Y$ is continuous prove that $f(X)$ is compact.
   (b) If $g\colon Y \to X$ is one-to-one, continuous and onto, must Y be compact? Justify your answer.

4. [duplicate of #3 of Fall 1998]

5. Suppose a polynomial is bounded by 1 in the unit disc. Show that all its coefficients are bounded by 1.

6. Let f be a holomorphic function on a neighbourhood of the annulus $A = \{1 \le |z| \le 2\}$. Suppose that $|f(z)| \le 1$ when $|z| = 1$ and that $|f(z)| \le \frac{1}{2}$ when $|z| = 2$. Show that $|zf(z)| \le 1$ on A.

7. Let C be the circle $|z-1| = 2$, traced *twice* in counterclockwise direction. Calculate the path integral
$$\int_C e^z(z^3+1)dz.$$

8. What type of singularity does $\cot(z)$ have at the origin? If it is a pole, find the order of the pole.

33. Fall 1998 (October)

1. (a) Is $f\colon [0,\infty) \to \mathbb{R}$ uniformly continuous? Justify your answers.

   (i) $f(x) = \sqrt{x}$;

   (ii) $f(x) = \sin(x^2)$;

   (iii) $f(x) = e^{-x}\sin(x^2)$.

   (b) Evaluate $\int_0^1 \left[\int_{x^2}^1 x^3 \cos(y^3)dy\right] dx$.

   (c) Use Green's theorem to evaluate $\int_\gamma Pdx + Qdy$ where $P = y + 3x$, $Q = 2y - x$, and γ is the ellipse $4x^2 + y^2 = 4$ traversed in the counterclockwise direction.

2. Let X be a metric space and let $\mathcal{O} = \{O_\alpha\}_{\alpha\in I}$ be an open cover of X. A real number $\lambda > 0$ is called a *Lebesgue number* for $\mathcal{O}$ iff every subset $Y \subseteq X$ whose diameter is less than λ must be contained in (at least) one O_α.

   (a) Prove that every open cover of a compact metric space has a Lebesgue number.

   (b) Find an open cover of $[1, 2) \subset \mathbb{R}$ that has no Lebesgue number.

3. Let X be compact and $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of continuous functions $X \to \mathbb{R}$. Suppose that $f\colon X \to \mathbb{R}$ is continuous, that $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in X$, and that $f_n \le f_{n+1}$ on X for all $n \in \mathbb{N}$. Prove that $f_n \to f$ uniformly on X.

4. (a) Suppose that $f \ge 0$ on $[a, b]$ and that $\int_a^b f(x)dx = 0$. Prove that $f = 0$ on $[a, b]$.

   (b) Suppose that f is Riemann integrable on $[a, b]$, that $D \subseteq [a, b]$ is dense, and that $g\colon X \to \mathbb{R}$ satisfies $f|_D = g|_D$.

   (i) If g is bounded on $[a, b]$, is g Riemann integrable on $[a, b]$?

   (ii) If g is Riemann integrable on $[a, b]$, must $\int_a^b f(x)dx = \int_a^b g(x)dx$?

34. Fall 1997 (September)

1. Let m, n be integers. Evaluate, with proof, the iterated limit
$$\lim_{n\to\infty}\left(\lim_{m\to\infty}|\cos(n!\pi x)|^m\right)$$
for each $x \in \mathbb{R}$.

2. Calculate
$$\int_0^\infty \frac{dx}{1+x^3}.$$

3. Find all solutions y, to the ordinary differential equation $x^3y'' + xy' = y$. Note $y = x$ is one solution.

4. Find the first four terms of the Laurent series of $e^z/(z(z^2+1))$ centred at zero. What is the largest open set on which it converges?

5. Suppose f is holomorphic on a neighbourhood of 0 satisfying $f(0) = 0 \neq f'(0)$. It has an inverse g defined on a neighbourhood of 0. (Do not prove this.). Show that there is an $\epsilon > 0$ and a neighbourhood U of 0 such that
$$g(z) = \frac{1}{2\pi i}\int_{|\xi|=\epsilon}\frac{f'(\xi)\xi}{f(\xi)-z} \text{ for } z \in U.$$

6. Let $I = (0,\infty)$ and $F\colon I^2 \to I^2$ be the map $(u,v) \mapsto (x,y) = (v(1+u)u, v(1+u)/u)$.
   (a) Show that F is differentiable and has differentiable inverse.
   (b) On a large, clearly labeled set of axes, sketch the region $F^{-1}((0,1)^2)$ and identify the curve $F^{-1}(\{(z,z) : z \in (0,1)\})$.
   (c) Evaluate the integral below by making the change of variables $(x,y) \mapsto (u,v)$
$$\int_0^1\int_0^y \frac{dx\,dy}{x+(xy)^{1/2}}.$$

7. Let $f\colon \mathbb{C}\setminus\{0\} \to \mathbb{C}\setminus\{0\}$ be a holomorphic bijection such that $f(1) = 1$. Show that either $f(z) = z$ or $f(z) = 1/z$ for all $z \in \mathbb{C}$.

35. Fall 1996 (October)

1. Show that there is a holomorphic function g on the domain $D = \mathbb{C} \setminus [-1, 1]$ such that
$$e^{g(z)} = \frac{z-1}{z+1}$$
for all $z \in D$.

2. Let A be a set, and suppose that $f : A \times A \to \mathbb{R}$ satisfies the following for all $a, b \in A$:
   - $f(a, b) = f(b, a) \geq 0$;
   - $f(a, b) = 0$ if and only if $a = b$.

   Define
$$d(a, b) = \inf \left\{ \sum_{i=1}^{n} f(a_{i-1}, a_i) : n \geq 0, a_i \in A, a_0 = n, a_n = b \right\}$$
   (a) Prove that d satisfies the triangle inequality on A, i.e., for all $a, b, c \in A$, one has
$$d(a, c) \leq d(a, b) + d(b, c).$$
   (b) Let $\mathbb{N} = \mathbb{Z}_{>0} = \{1, 2, \ldots\}$, and consider the function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ defined by
$$f(n, m) = \begin{cases} \frac{1}{n+m} & m \neq n \\ 0 & m = n. \end{cases}$$
   Show that d is not necessarily a metric on A.

3. Let γ be the unit circle in $\mathbb{C}$, positively oriented. Evaluate
$$\int_\gamma \frac{e^{1/z}}{z-2} dz$$

4. What is the radius of convergence of the Taylor series of the function
$$h(z) = \frac{e^{\sin(z)}}{(z+1+i)^2 \cos(z)}$$
centred at $z = 0$?

5. Find the family of curves in the plane orthogonal to the family $\{y = ax^3 : a \in \mathbb{R}\}$.

6. Let u be a real-valued, harmonic function in an open, connected subset of $\mathbb{R}^2$. Show that if u^2 is also harmonic, then u is constant.

   (Recall $u(x, y)$ is harmonic in an open set $U \subseteq \mathbb{R}^2$ iff, $\forall (x, y) \in U$, the following holds:
$$\frac{\partial^2 u}{(\partial x)^2}(x, y) + \frac{\partial^2 u}{(\partial y)^2}(x, y) = 0.)$$

7. Suppose $\phi_n : \mathbb{R} \to (0, \infty)$ satisfy
$$\int_\mathbb{R} \phi_n(t) dt = 1 \text{ for } n = 1, 2, \ldots; \quad \text{and} \quad \lim_{n\to\infty} \int_{|t|>\delta} \phi_n(t) dt = 0 \text{ for every } \delta > 0$$
   Let f be a bounded function on $\mathbb{R}$ which is continuous at x, and prove that
$$\lim_{n\to\infty} \int_\mathbb{R} \phi_n(x-t) f(t) dt = f(x)$$

8. Show that if an analytic function $f(z)$ has an essential singularity at a point p, then so does the function $\sin(f(z))$.

36. Fall 1994 (September)

36.1. **Part A.**

1. Study the convergence of the series

$$\sum_{n=1}^{\infty} \frac{x^{n+1}}{n(n+1)} \text{ for } x \in \mathbb{R},$$

and where it is convergent, find its sum. (P.S. Do not forget the endpoints.)

2. Prove or disprove: if (X, d) is a metric space and $\bar{d}\colon X \times X \to \mathbb{R}$ is given by

$$\bar{d}(x, y) = \frac{d(x, y)}{1 + d(x, y)},$$

then $\bar{d}$ is a metric on X.

3. Find the simple closed curve C for which the value of the contour integral

$$\int_C (y^3 - y)dx - 2x^3 dy$$

is a maximum.

4. Prove or disprove: if E and F are connected subsets of a metric space X, then

   (a) $E \cup F$ is connected;

   (b) $E \cap F$ is connected;

   (c) If $f\colon X \to \mathbb{R}$ is continuous, then $f(E)$ is connected.

5. It is easy to guess the value of

$$\lim_{n\to\infty} \int_0^n \left(1 - \frac{x}{n}\right)^n e^{x/2} dx.$$

   (a) What is it?

   (b) Justify (rigorously) your deduction (state all "applicable" theorems).

37. Fall 1993 (November)

37.1. **Real Analysis.**

1. Let f' exist and be bounded for $x \in \mathbb{R}$. Prove that f is uniformly continuous on the real line.

2. Let K be compact and $f\colon K \to \mathbb{R}$ be continuous, and let $M \subseteq K$ be given by
$$M = \{\, x : f(x) \geq f(k) \text{ for } k \in K \,\}.$$
Show that M is a compact set.

3. The Bessel function of zero order may be defined by
$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{4^n (n!)^2}$$
Find its radius of convergence and show that $y = J_0(x)$ is a *solution* of the differential equation $xy' + y' + xy = 0$.

4. Using the definition of integrable functions, prove that if f is continuous on $[a, b]$, then f is Riemann integrable on $[a, b]$.

5. Let $a_1, a_2, \ldots \in \mathbb{R}_{\geq 0}$ be a sequence with $a_n \to 0$ monotonically. Show that $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} 2^n a_{2^n}$ converges.

6. A transformation $T\colon \mathbb{R}^m \to \mathbb{R}^n$ is said to be *distance decreasing* iff there is a constant $r \in [0, 1)$ satisfying
$$|T(p) - T(q)| \leq r \cdot |p - q| \text{ for } p, q \in \mathbb{R}^m.$$
Let T be any distance-decreasing transformation of the plane into itself. Prove that T leaves exactly one point of the plane fixed; that is, $T(p) = p$ has one and only one solution.

37.2. **Complex Analysis.**

7. Let $f\colon \mathbb{C} \to \mathbb{C}$ be a continuous function which is analytic off $[-1, 1]$. Show that f is an entire function.

8. Prove that in the disc $|z| \leq 1$, we have $|e^z - 1| \leq (e - 1)|z|$. [Taylor series]

9. Find a bilinear transformation (i.e., a Möbius transformation) such that
$$z_1 = 1, z_2 = i, z_3 = 0 \text{ map to } w_1 = 0, w_2 = -1, w_3 = -i \text{ respectively}$$

10. Evaluate the following line/path/contour integrals:
    (a) $\frac{1}{2\pi i} \int \frac{\cos(z)+\sin(z)}{(z^2+25)(z+1)} dz$ around $\frac{x^2}{9} + \frac{y^2}{16} = 1$;
    (b) $\int \frac{dz}{e^z(z^2-1)}$ around the square with corners at $z = \pm 2$ and $z = \pm 2i$.

11. Let C be a closed rectifiable curve encircling the origin and $n \in \mathbb{Z}_{\geq 1}$. Show that
$$\frac{1}{2\pi} \int_C \left(z + \frac{1}{z}\right)^n dz = \begin{cases} \frac{n!}{(\frac{n-1}{2})!(\frac{n+1}{2})!} & n \text{ odd} \\ 0 & n \text{ even.} \end{cases}$$

12. Show that
$$\int_0^{2\pi} \frac{1 - a\cos\theta}{1 - 2a\cos\theta + a^2} d\theta = 2\pi \text{ for } a \in (-1, 1)$$